

A COUPLING, AND THE DARLING–ERDŐS CONJECTURES

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ABSTRACT. We derive a new coupling of the running maximum of an Ornstein–Uhlenbeck process and the running maximum of an explicit i.i.d. sequence. We use this coupling to verify a conjecture of Darling and Erdős (1956).

1. INTRODUCTION AND MAIN RESULTS

Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of independent, identically distributed random variables with $E[\xi_1] = 0$ and $E[\xi_1^2] = 1$, and define

$$(1.1) \quad U_n := \max_{1 \leq k \leq n} \left(\frac{\xi_1 + \dots + \xi_k}{k^{1/2}} \right) \quad (n \geq 1).$$

According to the law of the iterated logarithm (LIL) of Hartman and Wintner (1941),

$$(1.2) \quad \limsup_{n \rightarrow \infty} \frac{U_n}{(2 \log \log n)^{1/2}} = 1 \quad \text{almost surely.}$$

Here and throughout, “ $\log x$ ” and/or “ $\log(x)$ ” act as short-hand for “ $\ln(x \vee e)$.” In a remarkable paper (1956), Darling and Erdős establish the following variation of the LIL.

Theorem 1.1. *Assume that $E[\xi_1] = 0$, $E[\xi_1^2] = 1$, and $E\{|\xi_1|^3\} < \infty$. Then for all real numbers x ,*

$$(1.3) \quad \begin{aligned} \lim_{n \rightarrow \infty} P\{a(n)U_n - b(n) \leq x\} &= \exp\left(-\frac{e^{-x}}{(4\pi)^{1/2}}\right), \quad \text{where} \\ a(x) &:= (2 \log \log x)^{1/2}, \quad \text{and} \quad b(x) := 2 \log \log x + \frac{1}{2} \log \log \log x. \end{aligned}$$

Subsequently, Oodaira (1976) and Shorack (1979) improved the integrability condition to “ $E\{|\xi_1|^{2+\epsilon}\} < \infty$ for some $\epsilon > 0$.” The definitive result, along these lines, is due to Einmahl (1989) who improved the integrability condition further to “ $E\{\xi_1^2 \log \log |\xi_1|\} < \infty$,” and proved that the said condition is optimal. Related works can be found in Bertoin (1998) and Einmahl and Mason (1989).

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At the very end of their paper, Darling and Erdős pose two conjectures about strong-limit analogues of Theorem 1.1. Define

$$(1.4) \quad \begin{aligned} c_1 &:= \limsup_{n \rightarrow \infty} \frac{a(n)U_n - b(n)}{\log \log \log \log n}, \\ c_2 &:= -\liminf_{n \rightarrow \infty} \frac{a(n)U_n - b(n)}{\log \log \log \log n}. \end{aligned}$$

By the Kolmogorov zero-one law, c_1 and c_2 are constants almost surely.

The Darling–Erdős Conjecture. *With probability one, $0 < c_1, c_2 < \infty$.*

The main goal of this paper is to prove that $c_1 = \infty$ and $c_2 = 1$, under a mild moment condition on the distribution of ξ_1 . Thus, half of the conjecture is true while the other half is false.

Our proof involves first deriving a new and novel coupling (Theorem 2.1) of the running maximum of an Ornstein–Uhlenbeck process with the running maximum of a certain i.i.d. process. Theorem 1.1 follows readily from this coupling. Our solution to the Darling–Erdős conjecture also follows from it, but requires a little more work. En route, we present also an integral test (Theorem 3.1) for the lower envelope of the Ornstein–Uhlenbeck process; see also our integral test for pure-cosine lacunary series (Theorem 1.4). We adapt Breiman’s terminology (1968), and assert that our integral tests are “very delicate.” See Remark 3.2 below for an explanation.

Next we present the precise form of our solution to the Darling–Erdős conjecture.

Theorem 1.2. *With probability one, $c_1 = \infty$ and $c_2 = 1$, as long as*

$$(1.5) \quad E[\xi_1] = 0, \quad E[\xi_1^2] = 1, \quad \text{and } E[\xi_1^2 \log \log |\xi_1|] < \infty.$$

It turns out that the seemingly more interesting contradictory half (i.e., $c_1 = \infty$) is, in fact, not very deep. It relies only on results that were known prior to the work of Darling and Erdős (Erdős, 1942; Feller, 1946), and does not require any further computations. On the other hand, developing the formula “ $c_2 = 1$ ” seems to require some new ideas.

Shorack (1979) has observed that the strong invariance principles of Philipp and Stout (1975) yield Darling–Erdős theorems for many processes with dependent increments as well. Next, we explore this observation further in the special case of lacunary pure-cosine series only. It is both possible and tempting to use the constructions of Philipp and Stout (1975, Sections 3–12) and Berkes (1975), and find further embellishments (lacunary series with weights), and other applications (functions of strongly mixing random variables, partial sums of stationary Gaussian processes with long-range dependence, Markov sequences, etc.). However, we will not do that because no further ideas are needed to carry out such a program. Thus, we complete the Introduction by stating some implications of the present work in the context of pure-cosine series of the lacunary type.

Let $\{n_k\}_{k=1}^\infty$ be a sequence of real numbers that satisfies the *lacunarity condition*,

$$(1.6) \quad \liminf_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} > 1.$$

Consider the pure-cosine series,

$$(1.7) \quad f_n(\omega) = 2^{1/2} \sum_{1 \leq k \leq n} \cos(2\pi n_k \omega),$$

where $n \geq 1$ and $0 \leq \omega \leq 1$. Shorack (1979) has proved that the Darling–Erdős Theorem 1.1 continues to hold if we replace U_n by $\max_{1 \leq k \leq n} (f_k/k^{1/2})$, and the measure P by the Lebesgue measure on $[0, 1]$. More precisely, Shorack's theorem asserts that for all real numbers x , the following holds:

$$(1.8) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \left| \left\{ 0 \leq \omega \leq 1 : a(n) \max_{1 \leq k \leq n} \left(\frac{f_k(\omega)}{k^{1/2}} \right) - b(n) \leq x \right\} \right| \\ &= \exp \left(-\frac{e^{-x}}{(4\pi)^{1/2}} \right), \end{aligned}$$

where $|\cdot \cdot \cdot|$ denotes the Lebesgue measure. Shorack's proof rests on a strong approximation theorem of Philipp and Stout (1975, Theorem 3.1, p. 12) and a verification of two asymptotic negligibility conditions (Shorack, 1979, eq.'s (2.1) and (2.3)). It is possible, and not too hard, to replace the said asymptotic negligibility conditions with an appeal to the Erdős–Feller integral test, and use the strong approximation theorem of Berkes (1975) to prove the following.

Proposition 1.3. *Equation (1.8) continues to hold if the lacunarity condition (1.6) is replaced by the following weaker hypothesis:*

$$(1.9) \quad \text{There exists } 0 \leq \alpha < \frac{1}{2} \text{ such that } \lim_{k \rightarrow \infty} k^\alpha \left(\frac{n_{k+1}}{n_k} - 1 \right) = \infty.$$

[Mimic the argument the begins with (4.3) below.]

We obtain a corresponding integral test as an immediate consequence of the analysis of the present paper. Before we describe the said integral test let us define

$$(1.10) \quad \mathcal{F}_n(\omega) := a(n) \max_{1 \leq k \leq n} \left(\frac{f_k(\omega)}{k^{1/2}} \right) - b(n) + \frac{\log(4\pi)}{2},$$

where $n = 1, 2, \dots$ and $0 \leq \omega \leq 1$. The following is an immediate consequence of the strong approximation theorem of Berkes (1975) and our Theorem 3.1 below.

Theorem 1.4. *Suppose $\{n_k\}_{k=1}^\infty$ satisfies (1.9) and $h : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and satisfies $h(n+1) - h(n) = o(1/n)$ as $n \rightarrow \infty$. Then,*

$$(1.11) \quad |\{0 \leq \omega \leq 1 : \mathcal{F}_n(\omega) \leq -h(n) \text{ i.o.}\}| = \begin{cases} 0 & \text{if } \mathbf{J}(h) = \infty, \\ 1 & \text{if } \mathbf{J}(h) < \infty. \end{cases}$$

Here, “i.o.” means “infinitely often,” and

$$(1.12) \quad \mathbf{J}(h) := \int_1^\infty \exp\left(h(t) - e^{h(t)}\right) \frac{dt}{t}.$$

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2. A COUPLING FOR OU PROCESSES

Throughout, $X := \{X_t\}_{t \geq 0}$ denotes the Ornstein–Uhlenbeck (OU) process. We recall that X is a continuous, centered, Gaussian process with $\text{Cov}(X_s, X_t) = \exp(-|t-s|/2)$ for $s, t \geq 0$. We recall also that X is a stationary and ergodic diffusion, and X_0 is standard normal.

Let $\ell := \{\ell_t\}_{t \geq 0}$ denote the local times of X at zero. It is well known that ℓ is continuous (a.s.), and

$$(2.1) \quad \ell_t := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{\{0 < X_s < \epsilon\}} ds \quad (t \geq 0).$$

The convergence holds almost surely and in $L^p(\mathbf{P})$ for all $p \in [1, \infty)$. Define

$$(2.2) \quad \tau(t) := \inf\{s > 0 : \ell_s > t\} \quad (t \geq 0).$$

Also, we introduce the process $M := \{M_n\}_{n=1}^\infty$ as follows:

$$(2.3) \quad M_n := \sup_{\tau(n-1) \leq s \leq \tau(n)} X_s \quad \text{for all } n \geq 1.$$

By the strong Markov property, M is an i.i.d. sequence. Also, the distribution of M has been computed explicitly in Proposition 2.2 of Khoshnevisan, Levin, and Shi (2005). It reads as follows: For all real numbers x ,

$$(2.4) \quad \mathbf{P}\{M_1 \leq x\} := F(x) := \exp\left(-\frac{1}{2 \int_0^{\max(x,0)} \exp(y^2/2) dy}\right),$$

where $\exp(-1/0) := 0$. Now we present and prove the following coupling.

Theorem 2.1. *As $t \rightarrow \infty$,*

$$(2.5) \quad \mathbf{P}\left\{\sup_{0 \leq s \leq t} X_s \neq \max_{1 \leq j \leq (2\pi)^{-1/2}t} M_j\right\} = O\left((\log(t)/t)^{1/2}\right).$$

Our proof of Theorem 2.1 requires a technical lemma. But first, let us observe from (2.1) that $\mathbf{E}[\ell(t)] = t(2\pi)^{-1/2}$. Thus, by the ergodic theorem, $\lim_{t \rightarrow \infty} \ell(t)/t = (2\pi)^{-1/2}$ a.s. A time substitution then yields the following:

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{\tau(t)}{t} = (2\pi)^{1/2} \quad \text{a.s.}$$

Convergence holds also in $L^p(\mathbf{P})$ for all $p \in [1, \infty)$; confer with (2.11) below. The aforementioned technical lemma is the following quantitative refinement of the ergodic theorem (2.6).

Proposition 2.2. *For all $\alpha > 0$ there exists $\beta > 1$ such that for all $t \geq 1$,*

$$(2.7) \quad \mathbf{P} \left\{ \sup_{0 \leq s \leq t} \left| \tau(s) - s(2\pi)^{1/2} \right| \geq \beta(t \log t)^{1/2} \right\} \leq \beta t^{-\alpha}.$$

Proof. The functions τ , $f(x) = x$, and $g(x) = (x \log x)^{1/2}$ are non-decreasing. Therefore, it suffices to prove that for all $\alpha > 0$ there exists $\beta > 1$ such that for all integers $n \geq 1$,

$$(2.8) \quad \mathbf{P} \left\{ \max_{1 \leq i \leq n} \left| \tau(i) - i(2\pi)^{1/2} \right| \geq \beta(n \log n)^{1/2} \right\} \leq \beta n^{-\alpha}.$$

By the strong Markov property, τ is an ordinary random walk. This, (2.6), and the Kolmogorov strong law of large numbers together imply that $\mathbf{E}[\tau(1)] = (2\pi)^{1/2}$. We propose to verify that $\tau(1)$ has a finite moment generating function. Then, (2.8) follows at once from the classical moderate deviations estimates of Cramér (Ibragimov and Linnik, 1971, Theorem 6.1.1, p. 156).

Define $T_1 := \inf\{s > 0 : X_s = 1\}$, $T'_1 := \inf\{s > T_1 : X_s = 0\}$, $T_2 := \inf\{s > T'_1 : X_s = 1\}$, etc. These are the respective crossing-times of one and zero.

Because the speed measure of X decays faster than exponentially, T_1 has exponential moments of all order; i.e., $\mathbf{E}_x[\exp(aT_1)] < \infty$ for all $a > 0$ and $x \in (-\infty, \infty)$ (Mandl, 1968, Lemma 2, p. 112). Furthermore, if $T_0 := 0$, then the strong Markov property of X guarantees that:

- (1) $\{(T_n - T_{n-1}, \ell(T_n) - \ell(T_{n-1}))\}_{n=2}^\infty$ is an independent sequence;
- (2) The \mathbf{P} -distribution of $T_n - T_{n-1}$ is the same as the \mathbf{P}_1 -distribution of T_1 ;
- (3) The \mathbf{P} -distribution of $\ell(T_n) - \ell(T_{n-1})$, and the \mathbf{P}_1 -distribution of $\ell(T_1)$, are the same, in fact exponential.

Define $\mu := \mathbf{E}_1[\ell(T_1)]$, and note the bounds,

$$(2.9) \quad \begin{aligned} \mathbf{P}\{\tau(1) > T_n\} &= \mathbf{P}\{\ell(T_n) < 1\} \\ &\leq \mathbf{P} \left\{ \sum_{2 \leq i \leq n} (\ell(T_i) - \ell(T_{i-1})) < 1 \right\} \\ &\leq \mathbf{P} \left\{ \sum_{2 \leq i \leq n} (\ell(T_i) - \ell(T_{i-1})) \leq \frac{n-1}{2}\mu \right\}, \end{aligned}$$

for all n large enough. By large deviations, this implies the existence of a constant c such that for all n large, $\mathbf{P}\{\tau(1) > T_n\} \leq \exp(-cn)$; see, for example Ibragimov and Linnik (1971, eq. 13.2.1, p. 245). Thus, for all n

large, letting $\nu := \mathbb{E}_1[T_1]$,

$$(2.10) \quad \mathbb{P}\{\tau(1) > 2\nu n\} \leq \exp(-cn) + \mathbb{P}\{T_n \geq 2\nu n\}.$$

We can write $T_n = (T_n - T_{n-1}) + \cdots + (T_2 - T_1) + T_1$. We recall that T_1 and $T_i - T_{i-1}$ are independent, and the $(T_i - T_{i-1})$'s are i.i.d. Moreover, all have finite exponential moments of all orders. Therefore, another appeal to large deviations proves that $\mathbb{P}\{T_n > 2\nu n\} \leq \exp(-c'n)$ for some constant $c' > 0$ that does not depend on n . From this we can conclude that

$$(2.11) \quad \mathbb{E}[\exp(a\tau(1))] < \infty \quad \text{for some } a > 0.$$

As was mentioned earlier, (2.8) follows from this at once. This proves the proposition. \square

Now we can prove Theorem 2.1

Proof of Theorem 2.1. Define $E_\beta(t)$ to be complement of the event,

$$(2.12) \quad \left\{ \max_{1 \leq j \leq t-\beta(t \log t)^{1/2}} M_j \leq \sup_{0 \leq s \leq t(2\pi)^{1/2}} X_s \leq \max_{1 \leq j \leq t+\beta(t \log t)^{1/2}} M_j \right\}.$$

For all integers $n, m \geq 1$,

$$(2.13) \quad \mathbb{P}\left\{ \max_{1 \leq j \leq n} M_j \neq \max_{1 \leq i \leq n+m} M_i \right\} = \frac{m}{n+m}.$$

Therefore, as $t \rightarrow \infty$,

$$\begin{aligned} & \mathbb{P}\left\{ \sup_{0 \leq s \leq t(2\pi)^{1/2}} X_s \neq \max_{1 \leq j \leq t} M_j \right\} \\ (2.14) \quad & \leq \mathbb{P}(E_\beta(t)) + \mathbb{P}\left\{ \max_{1 \leq j \leq t-\beta(t \log t)^{1/2}} M_j \neq \max_{1 \leq j \leq t+\beta(t \log t)^{1/2}} M_j \right\} \\ & = \mathbb{P}(E_\beta(t)) + (2\beta + o(1)) \left(\frac{\log t}{t}\right)^{1/2}. \end{aligned}$$

Thanks to Proposition 2.2, we can choose and fix β , once and for all, so large that the probability of $E_\beta(t)$ is $O((\log(t)/t)^{1/2})$. This completes our proof. \square

3. AN INTEGRAL TEST FOR OU PROCESSES

By (2.4) and a direct computation,

$$(3.1) \quad \lim_{n \rightarrow \infty} \mathbb{P}\left\{ \max_{1 \leq j \leq n} M_j \leq (2 \log n + \log \log n + x)^{1/2} \right\} = \exp\left(-\frac{e^{-x/2}}{2^{1/2}}\right),$$

for every real number x . Our coupling (Theorem 2.1) then yields the following without further effort: For all real numbers x ,

$$(3.2) \quad \lim_{t \rightarrow \infty} \mathbb{P}\left\{ a(e^t) \sup_{0 \leq s \leq t} X_s - b(e^t) \leq x \right\} = \exp\left(-\frac{e^{-x}}{(4\pi)^{1/2}}\right).$$

This is the analogue of the Darling–Erdős theorem for OU processes, and is implicitly the first part of the original proof of Theorem 1.1. Explicitly, it appears as a special case of a result of Pickands (1967, Theorem 4.4) for Gaussian processes. Earlier, Newell (1962, pp. 491–492), studying diffusions, obtained a version of (3.2) asymptotic in x . It also can be found in Shorack (1979), and as a consequence of a much more general theorem of Bertoin (1998, Theorem 3). There is extensive literature on the maximum of stationary Gaussian processes; for a sampling see Volkonskii and Rozanov (1961), Cramér (1965), Qualls and Watanabe (1972), and Berman (1992).

The main purpose of this section is to derive an integral test that corresponds to the \liminf behavior of $a(e^t) \sup_{0 \leq s \leq t} X_s - b(e^t)$. We find it more convenient to work with the following variant:

$$(3.3) \quad \mathcal{X}(t) := a(e^t) \sup_{0 \leq s \leq t} X_s - b(e^t) + \frac{\log(4\pi)}{2} \quad (t > 0).$$

Suppose g is a Borel-measurable function that is non-decreasing ultimately, and has the following additional properties:

$$(3.4) \quad \frac{\log(g(n+1))}{\log(g(n))} = 1 + o(1/n) \quad (n \rightarrow \infty).$$

Also define for all measurable functions $g : [1, \infty) \rightarrow (0, \infty)$,

$$(3.5) \quad \mathbf{I}(g) := \int_1^\infty \frac{\log(g(t))}{tg(t)} dt.$$

To compare with (1.12) we note merely that $\mathbf{I}(g) = \mathbf{J}(\log \log g)$. Then we have the following:

Theorem 3.1. *Assume g satisfies (3.4), and define F to be the event that the random set $\{t > 0 : \mathcal{X}(t) \leq -\log \log(g(t))\}$ is unbounded. Then $P(F) = 0$ or 1 according as $\mathbf{I}(g) = \infty$ or $\mathbf{I}(g) < \infty$.*

Remark 3.2. This is a “very delicate” LIL (Breiman, 1968) in the following sense: If we perturb the gauge function $-\log \log(g(t))$ even a little and replace it by $-\log(c + \log(g(t)))$, then the end-result could be vastly different. For example, it follows from Theorem 3.1 that for all $\epsilon > 0$, a.s.:

$$(3.6) \quad \begin{aligned} \mathcal{X}(t) &\leq -\log(\log \log t + 2 \log \log \log t) \text{ unboundedly, whereas} \\ \mathcal{X}(t) &> -\log(\log \log t + (2 + \epsilon) \log \log \log t) \text{ eventually.} \end{aligned}$$

The delicateness of the integral test is now seen, for the difference between the right-most terms in (3.6) is $(\epsilon + o(1))(\log \log \log t)(\log \log t)^{-1} = o(1)$ as $t \rightarrow \infty$, whereas $\lim_{t \rightarrow \infty} \mathcal{X}(t) = -\infty$ a.s.

We will prove that Theorem 3.1 is a consequence of two technical lemmas. Those are developed first. Throughout, g is a Borel-measurable function that is non-decreasing ultimately.

Lemma 3.3. *If g satisfies (3.4), then $\mathbf{I}(g) < \infty$ if and only if almost surely for all but a finite number of n 's,*

$$(3.7) \quad \max_{1 \leq j \leq n} M_j > (2 \log n + \log \log n - \log(2) - 2 \log \log(g(n)))^{1/2}.$$

Proof. We can assume, without loss of generality, that

$$(3.8) \quad \log n \leq g(n) \leq \log n \cdot (\log \log n)^3 \quad (n \geq 1).$$

For otherwise we could replace g everywhere by g_1 , where

$$(3.9) \quad g_1(x) := \min \{ \max(g(x), \log x), \log x \cdot (\log \log x)^3 \}.$$

Recall F from (2.4), and define $\bar{F} := 1 - F$. Let $\{u_n\}_{n=1}^\infty$ be a non-decreasing sequence of positive real numbers such that $n\bar{F}(u_n)$ is also non-decreasing. Then, according to Theorem 4.3.1 of Galambos (1978, p. 214),

$$(3.10) \quad P \left\{ \max_{1 \leq j \leq n} M_j < u_n \text{ i.o.} \right\} = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty \bar{F}(u_n) e^{-n\bar{F}(u_n)} < \infty, \\ 1 & \text{if } \sum_{n=1}^\infty \bar{F}(u_n) e^{-n\bar{F}(u_n)} = \infty. \end{cases}$$

In fact, the monotonicity of u_n and $n\bar{F}(u_n)$ can be replaced by the following condition, as can be seen by inspecting the proofs in Galambos (1978, pp. 214–222):

$$(3.11) \quad u_k \text{ is ultimately increasing, and } \max_{1 \leq k \leq n} k\bar{F}(u_k) = n\bar{F}(u_n) + O(1),$$

as $n \rightarrow \infty$. A little bit of calculus shows that

$$(3.12) \quad \left| \int_0^x \exp(y^2/2) dy - \frac{1}{x} e^{x^2/2} \right| = O \left(x^{-3} e^{-x^2/2} \right) \quad (x \rightarrow \infty).$$

Therefore, by Taylor's expansion,

$$(3.13) \quad \left| \bar{F}(x) - \frac{1}{2} x e^{-x^2/2} \right| = O \left(x^{-3} e^{-x^2/2} \right) \quad (x \rightarrow \infty).$$

We apply the preceding with

$$(3.14) \quad u_n := (2 \log n + \log \log n - \log(2) - 2 \log \log(g(n)))^{1/2} \quad (n \geq 1).$$

Then, (3.4) and (3.8) together imply that

$$(3.15) \quad \sum_{n \geq 1} \bar{F}(u_n) e^{-n\bar{F}(u_n)} < \infty \iff \mathbf{I}(g) < \infty.$$

We omit the details as they involve routine computations. Similar work shows that we have (3.11). Whence follows the lemma. \square

We apply Theorem 2.1 and Lemma 3.3 in conjunction to obtain the following.

Lemma 3.4. *Under the preceding conditions, $\mathbf{I}(g) < \infty$ if and only if almost surely for all but a bounded set of t 's,*

$$(3.16) \quad \sup_{0 \leq s \leq t} X_s > (2 \log t + \log \log t - \log(4\pi) - 2 \log \log(g(t)))^{1/2}.$$

Proof. Let $m_n := [\exp(\tau n / \log n)]$ where $\tau \in (0, 1)$ is small enough but otherwise fixed; see the top of page 222 of Galambos (1978) for details.

Thanks to Galambos (1978, pp. 218–222), our proof of Lemma 3.3 implies, in fact, that $\mathbf{I}(g) < \infty$ if and only if almost surely for all but a finite number of n 's,

$$(3.17) \quad \begin{aligned} & \max_{1 \leq j \leq m_n} M_j \\ & > (2 \log m_n + \log \log m_n - \log(2) - 2 \log \log(g(m_n)))^{1/2}, \end{aligned}$$

and this is, in turn, equivalent to the validity of the following for all but a finite number of n 's:

$$(3.18) \quad \begin{aligned} & \max_{1 \leq j \leq m_{n-1}} M_j \\ & > (2 \log m_n + \log \log m_n - \log(2) - 2 \log \log(g(m_n)))^{1/2}, \end{aligned}$$

But according to Theorem 2.1 and the Borel–Cantelli lemma, with probability one,

$$(3.19) \quad \max_{1 \leq j \leq m_n} M_j = \sup_{0 \leq s \leq m_n (2\pi)^{1/2}} X_s \quad \text{eventually as } n \rightarrow \infty.$$

This, and monotonicity, together prove the lemma. \square

Proof of Theorem 3.1. As in the proof of Lemma 3.3, we assume without loss of generality that (3.8) holds. Next we make some real-variable computations.

For all $\epsilon > 0$ small enough,

$$(3.20) \quad (1 + \epsilon)^{1/2} \leq 1 + \frac{\epsilon}{2} \leq (1 + \epsilon + \epsilon^2)^{1/2}.$$

Choose and fix a real number p and define

$$(3.21) \quad \epsilon(t) := \frac{\log \log t}{2 \log t} + \frac{p}{2 \log t} - \frac{\log \log(g(t))}{\log t}.$$

Plug in $\epsilon := \epsilon(t)$ in the first bound in (3.20) to deduce that

$$(3.22) \quad \begin{aligned} & \left(1 + \frac{\log \log t}{2 \log t} + \frac{p}{2 \log t} - \frac{\log \log(g(t))}{\log t} \right)^{1/2} \\ & \leq 1 + \frac{\log \log t}{4 \log t} + \frac{p}{4 \log t} - \frac{\log \log(g(t))}{2 \log t}. \end{aligned}$$

Condition (3.8) implies that

$$(3.23) \quad \epsilon^2(t) = O\left(\left(\frac{\log \log t}{\log t}\right)^2\right) = o\left(\frac{1}{\log t \cdot \log(g(t))}\right) \quad (t \rightarrow \infty).$$

Now choose and fix a number $c \in (0, 1)$, and apply (3.4) once again to find that

$$(3.24) \quad 0 \leq \frac{\log \log(g(t)) - \log \log(cg(t))}{\log t} = \frac{\log(1/c) + o(1)}{\log t \cdot \log(g(t))} \quad (t \rightarrow \infty).$$

The last two displays together with (3.20) imply that almost surely as t grows to infinity,

$$(3.25) \quad \left(\frac{\log \log t}{2 \log t} + \frac{p}{2 \log t} - \frac{\log \log(cg(t))}{\log t} \right)^{1/2} \geq 1 + \frac{\epsilon(t)}{2} \quad \text{eventually.}$$

The theorem follows readily from (3.22), (3.25), and Lemma 3.4, because $\mathbf{I}(g) < \infty$ if and only if $\mathbf{I}(cg) < \infty$. \square

4. PROOF OF THEOREM 1.2

We can recast Feller's (1946) improvement on the integral test of Erdős (1942) as follows: For any non-decreasing sequence $\{r_n\}_{n=1}^\infty$ of positive real numbers,

$$(4.1) \quad P\{U_n \leq r_n \text{ eventually}\} = 1 \quad \text{if and only if} \quad \sum_{n \geq 1} \frac{r_n}{n} e^{-r_n^2/2} < \infty.$$

Feller's test is valid solely under the condition (1.5); see Einmahl (1989) where a fatal gap in Feller's proof was bridged. Of course, if the probability in (4.1) is strictly less than one, then it is zero. This follows from the Kolmogorov zero-one law.

We can apply Feller's test to the sequence

$$(4.2) \quad r_n := (2 \log \log n + 3 \log \log \log n + \theta \log \log \log \log n)^{1/2}.$$

For this particular choice of r_n 's, the summability condition of (4.1) holds if and only if $\theta > 2$. A little algebra yields (5.1), whence follows that $c_1 = \infty$. Now we turn to proving that $c_2 = 1$, all the time assuming that (1.5) holds.

First of all, we note that

$$(4.3) \quad \liminf_{t \rightarrow \infty} \frac{a(t) \sup_{0 \leq s \leq \log t} X_s - b(t)}{\log \log \log t} = -1 \quad \text{almost surely.}$$

This is a consequence of Theorem 3.1, and was mentioned earlier in a slightly different form (Remark 3.2). From here on, we use strong approximations. The forthcoming argument is inspired by those of Oodaira (1976) and Shorack (1979).

Define $S_0 := \xi_0 := 0$ and $S_t := \sum_{0 \leq j \leq t} \xi_j$ as usual ($t > 0$). We can note that $H(t) := t^2 \log \log t$ satisfies the conditions of Theorem 2 of Einmahl (1987). Thus, Einmahl's theorem implies that we can construct $\{S_t\}_{t \geq 0}$ together with a (standard) Brownian motion $\{B_t\}_{t \geq 0}$ —on a suitably-chosen probability space—such that almost surely,

$$(4.4) \quad |S_s - B_s| = o\left(\left(\frac{s}{\log \log s}\right)^{1/2}\right) \quad (s \rightarrow \infty).$$

Define

$$(4.5) \quad c(n) := \exp\left(\frac{\log n}{(\log \log n)^3}\right) \quad (n \geq 1).$$

Choose and fix $\epsilon > 0$. According to (4.1), the following holds almost surely: For all n sufficiently large,

$$(4.6) \quad \begin{aligned} U_{c(n)} &\leq (2 \log \log c(n) + 3 \log \log \log n + (2 + \epsilon) \log \log \log \log n)^{1/2} \\ &\leq a(n) \left[1 - \frac{3 \log \log \log n}{2 \log \log n} + \frac{(1 + \epsilon) \log \log \log \log n}{\log \log n} \right]. \end{aligned}$$

Consequently,

$$(4.7) \quad \liminf_{n \rightarrow \infty} \frac{a(n)U_{c(n)} - b(n)}{\log \log \log \log n} = -\infty \quad \text{almost surely.}$$

Similarly, we have

$$(4.8) \quad \liminf_{n \rightarrow \infty} \frac{a(n) \sup_{1 \leq s \leq c(n)} (B_s / s^{1/2}) - b(n)}{\log \log \log \log n} = -\infty \quad \text{almost surely.}$$

On the other hand, thanks to (4.4), almost surely,

$$(4.9) \quad \sup_{c(n) \leq s \leq n} \frac{|B_s - S_s|}{s^{1/2}} = o\left((\log \log c(n))^{-1/2}\right).$$

Because $\log \log c(n) = (1 + o(1)) \log \log n$, it follows that almost surely,

$$(4.10) \quad \limsup_{t \rightarrow \infty} \frac{a(n)}{\log \log \log \log n} \sup_{c(n) \leq s \leq n} \frac{|S_s - B_s|}{s^{1/2}} = 0.$$

But the process $\{e^{-s/2} B_{\exp(s)}\}_{s \geq 0}$ has the same finite-dimensional distributions as X . Combine this observation with (4.3), (4.7), (4.8), and (4.10) to complete the proof.

5. EPILOGUE

5.1. Further Refinements. During the course of our proof of Theorem 1.2, we have proved the following facts:

- (1) If $E\{\xi_1^2 \log \log |\xi_1|\} < \infty$, then with probability one,

$$(5.1) \quad \limsup_{n \rightarrow \infty} \frac{(2 \log \log n)^{1/2} U_n - 2 \log \log n - \frac{3}{2} \log \log \log n}{\log \log \log \log n} = 1.$$

- (2) The moment condition (1.5) implies also that with probability one,

$$(5.2) \quad \liminf_{n \rightarrow \infty} \frac{(2 \log \log n)^{1/2} U_n - 2 \log \log n - \frac{1}{2} \log \log \log n}{\log \log \log \log n} = -1.$$

It is not hard to see from our arguments that more stringent moment conditions yield more detailed results. We leave the details to the interested reader.

5.2. Toward a Conjecture of Pickands. There are abundant techniques already available for handling the limit superior behavior of the running maxima of Gaussian and diffusion processes (Berman, 1992; Pickands, 1967; , 1969a; , 1969b; Qualls and Watanabe, 1971; , 1972). In comparison, the literature on limit inferior behavior is scant. The following noteworthy result along these lines is due to Pickands (1969b, Theorem 3.2):

$$(5.3) \quad \liminf_{t \rightarrow \infty} \frac{(2 \log t)^{1/2} \sup_{0 \leq s \leq t} X_s - 2 \log t}{\log \log t} \geq \frac{1}{2} \quad \text{almost surely.}$$

Our integral test (Theorem 3.1) improves this in a definitive manner. In particular, it follows that with probability one,

$$(5.4) \quad \liminf_{t \rightarrow \infty} \frac{(2 \log t)^{1/2} \sup_{0 \leq s \leq t} X_s - 2 \log t}{\log \log t} = \frac{1}{2}.$$

This verifies a conjecture of Pickands (1969b, p. 86) in the special case of OU processes. The methods of Pickands (1969b) have been applied to study similar problems in other settings (Pickands, 1969a). Our techniques are quite different, however, and do not seem to work when the process in question is not Markovian.

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